UDC 539.374
Doi: 10.31772/2712-8970-2021-22-3-452-456

Для цитирования: Сенашов С. И., Савостьянова И. Л. Пространственные двумерные решения // Сибирский аэрокосмический журнал. 2021. Т. 22, № 3. С. 452–456. Doi: 10.31772/2712-8970-2021-22-3-452-456.

For citation: Senashov S. I., Savostyanova I. L. 3-dimensional solutions from two variables. *Siberian Aerospace Journal*. 2021, Vol. 22, No. 3, P. 452–456. Doi: 10.31772/2712-8970-2021-22-3-452-456.

Пространственные двумерные решения

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В работе рассматриваются стационарные пространственные уравнения идеальной пластичности с условием текучести Мизеса. Материал предполагается несжимаемым. Подробно изучен случай, когда все три компоненты вектора скорости и гидростатическое давление зависят только от двух координат x, y. Для этого случая введено новое название – пространственная двумерная система уравнений, чтобы отличить ее от общепринятых двумерных систем уравнений, когда от нуля отличны только две компоненты вектора скорости и гидростатическое давление. Доказано, что система допускает, в смысле С. Ли, алгебру Ли размерности 10. Показано, что пространственное двумерное деформированное состояние – это есть суперпозиция плоского напряженного состояния и пластического кручения вокруг оси z. Построены два инвариантных решения уравнений, описывающих пространственное двумерное деформированное состояние. Первое решения можно использовать для описания пластических течений между двумя жесткими плитами, которые сближаются с разными скоростями. Второе решение служит для описания напряженно-деформированного состояния материала внутри плоского канала, образованного сходящимися плитами.

Ключевые слова: пространственные решения уравнений идеальной пластичности, точечные симметрии, инвариантные решения.

3-dimensional solutions from two variables

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In this paper, we consider stationary 3-dimensional equations of ideal plasticity with the Mises flow condition. The material is assumed to be incompressible. The case when all three components of the velocity vector and hydrostatic pressure depend only on two coordinates x, y is studied in detail. For this case, a new name is introduced – 3-dimensional solutions from two variables, to distinguish it from the generally accepted two-dimensional state, when only two components of the velocity vector and hydrostatic pressure differ from zero. It is proved that the system admits, in the sense of S. Lie, a Lie algebra of dimension 10. It is shown that all 3-dimensional solutions from two variables is a superposition of the plane stress state and *plastic torsion around the z-axis. Two invariant solutions of the equations describing the 3-dimensional deformed state are constructed. The first solution can be used to describe plastic flows between two rigid plates that approach at different speeds. The second solution is used to describe the stress-strain state of the material inside a flat channel formed by converging plates.*

Keywords: 3-dimensional solutions of ideal plasticity equations, point symmetries, invariant solutions.

Introduction

The notion of "3-dimensional solutions from two variables" has been added to the title of the article. Mechanics know the plane deformed state – this is the case when in a two-dimensional Cartesian coordinate system two components of the strain rate vector and the hydrostatic pressure depend on x, y. The plane stress state is also known – this is when the components of the stress tensor $\sigma_z, \tau_{xz}, \tau_{yz}$ are equal to zero, and the components $\sigma_x, \sigma_y, \tau_{xy}$ do not depend on z.

In our case, all components of the stress tensor do not depend on z, just such a case we called the spatial two-dimensional state.

The system of 3-dimensional equations of plasticity in the Cartesian coordinate system $x_1 = x$, $x_2 = y$, $x_3 = z$ in the stationary case has the form

$$\partial_j s_{ij} = \partial_i p, s_{ij} = \lambda\left(\partial_i u_j + \partial_i u_i\right)/2, \partial_i u_j = 0, s_{ij}s_{ij} = 2k_s^2, i, j = 1,2,3 \tag{1}$$

Here σ_{ij}, s_{ij} are components of the tensor and deviator of the stress tensor; $u_1 = u$, $u_2 = v$, $u_3 = w$ – components of the strain rate vector; λ – non-negative function; k_s – constant plasticity; p – the hydrostatic pressure, summation is carried out over repeated indices.

Eliminating the stress tensor deviator components from the system of equations (1), we obtain the following nonlinear system of equations

$$\partial_i p = \frac{\sqrt{2}k}{2A}\partial_{jj}^2 u_i - \frac{\sqrt{2}k}{A^3} e_{ij} e_{mn} \partial_{mj}^2 u_n,\ e_{ij} e_{ij} = A^2, \partial_i u_i = 0. \tag{2}$$

It is known that the system of equations (2) is of elliptic type. Let us describe the known solutions of this system.

The solutions of this system were constructed by R. Hill in 1948 [1], W. Prager in 1954 [2], D. D. Ivlev in 1960 [3; 4], MA Zadoyan in 1964 [5–8], as well as by the authors of this article [9–13]. We also note a number of exact solutions constructed by B. D. Annin [14] for the plasticity equations in the spatial case with the Tresk yield condition.

Symmetries of system (2)

The group of point symmetries of the system of equations (2) is generated by the following operators

$$X_i = \partial_{x_i},\ Y_i = \partial_{u_i},\ N = x_i \partial_{x_i},\ M = u_i \partial_{u_i},\ i = 1,2,3.$$

$$T_1 = x_2 \partial_{u_3} - x_3 \partial_{u_2},\ T_2 = x_3 \partial_{u_1} - x_1 \partial_{u_3},\ T_3 = x_1 \partial_{u_2} - x_2 \partial_{u_1},$$

$$Z_1 = x_2 \partial_{x_3} - x_3 \partial_{x_2} + u_2 \partial_{u_3} - u_3 \partial_{u_2},\ Z_2 = x_3 \partial_{x_1} - x_1 \partial_{x_3} + u_3 \partial_{u_1} - u_1 \partial_{u_3}, \tag{3}$$

$$Z_3 = x_1 \partial_{x_2} - x_2 \partial_{x_1} + u_1 \partial_{u_2} - u_2 \partial_{u_1},\quad S = \partial_p.$$

The operators X_i, Y_i, N, M correspond to the following continuous transformations

$$x'_i = x_i + a_i,\ \ u'_i = u_i + b_i,\ \ x'_i = x_i \exp a,\ \ u'_i = u_i \exp b,\ \ i = 1,2,3.$$

These are translations along the coordinates and components of the strain rate vector, as well as tension.

Transformations T_i show that the system of equations (2) does not change under rigid displacements

$$u'_2 = u_2 - c_1 x_3,\ \ u'_3 = u_3 + c_1 x_2,\ \ u'_1 = u_1 + c_2 x_3,$$
$$u'_3 = u_3 - c_2 x_1,\ \ u'_2 = u_2 + c_3 x_1,\ \ u'_1 = u_1 - c_3 x_2.$$

The groups generated by the operators Z_i, are rotations around the coordinate axes

$$x'_2 = x_2 \cos\varphi_1 + x_3 \sin\varphi_1,\ \ x'_3 = -x_2 \sin\varphi_1 + x_3 \cos\varphi_1,$$
$$x'_3 = x_3 \cos\varphi_2 + x_1 \sin\varphi_2,\ \ x'_1 = -x_3 \sin\varphi_2 + x_1 \cos\varphi_2,$$
$$x'_1 = x_1 \cos\varphi_3 + x_2 \sin\varphi_3,\ \ x'_2 = -x_1 \sin\varphi_3 + x_2 \cos\varphi_3.$$

The last transformation describes the invariance of the hydrostatic pressure with respect to displacements

$$p' = p + d.$$

In all these formulas a_i, b_i, c_i, φi, d are group parameters. It is usually assumed that they vary continuously in a neighbourhood of zero.

The investigations carried out have shown that all the solutions constructed by R. Hill, V. Prager, D. D. Ivlev, and M. A. Zadoyan are invariant solutions with respect to some one-dimensional subgroups of point transformations generated by operators (3). The invariance here means that the solutions do not change under some transformations generated by the symmetry group (3). Thus, R. Hill's solution is invariant under the subalgebra generated by the operator $2C_0 S + X_1 + \alpha Y_1 + \beta T_1$, D. D. Ivlev's solution is invariant under the subalgebra $2C_0 S + X_1 + \alpha Y_1$, Prager's solution is invariant under the subalgebra $aS + X_1 + T_1 + \alpha T_2$, Zadoyan's solution is invariant under the same subalgebra. What does this fact mean? It says that in fact all these solutions are "two-dimensional", that is, in a suitable coordinate system they can be written as functions of only two independent variables. The same can be said about the solutions constructed by the authors of this work. Then the question arises: what is to be understood by a 3-dimensional solution? Based on the solutions presented here, the answer is as follows: 3-dimensional solutions are solutions that have three components of the velocity vector, pressure, which actually depend on two variables in a suitable coordinate system. These solutions are invariant solutions of rank 2. In this case, the problem of finding 3-dimensional solutions can be formally stated as follows: construction of new invariant solutions of rank 2 for 3-dimensional equations of ideal plasticity. The form of such solutions can be easily enumerated if we enumerate all the different, up to inner automorphisms, one-dimensional subalgebras of the algebra (3).

There are several subalgebras on which invariant rank 2 solutions can be constructed. Let's list them.

$$\begin{aligned}
&X_3 + \gamma S, X_1 + Z_1 + \gamma S, \alpha M + N + \gamma S, N + Y_1 + \gamma S,\\
&Z_1 + \alpha N + Y_1 + \gamma S, Z_1 + \alpha N + \beta M + \gamma S,\\
&X_1 + \alpha Z_1 + M + \gamma S,\\
&X_1 + Y_1 + \alpha T_1 + \gamma S,\ X_1 + Z_1 + T_1 + \gamma S, M + N + T_1 + \gamma S,\\
&Z_1 + Y_1 + \alpha T_1 + \gamma S, X_1 + \alpha X_2 + T_2 + \beta T_3 + \gamma S,\\
&X_1 + Z_1 + Y_1 + \alpha T_1 + \gamma S.
\end{aligned} \qquad (4)$$

Here α, β, γ are arbitrary constants, different values of these constants correspond to dissimilar subalgebras.

In this paper, we consider only solutions invariant with respect to the subalgebra $X_3 + \gamma S$.

Note. Other solutions based on subalgebras (4), as well as the form of all invariant solutions that can be constructed for the system of equations (1), can be found in [11].

Solutions invariant with respect to this subalgebra should be sought in the form

$$u = u(x,y),\ v = v(x,y),\ w = w(x,y),\ p = p(x,y) + \gamma z. \tag{5}$$

Substituting relations (5) into the system of equations (1), we obtain

$$\begin{gathered}
\partial_x \sigma_x + \partial_y \tau_{xy} = 0,\ \partial_x \tau_{xy} + \partial_y \sigma_y = 0,\ \partial_x \tau_{xz} + \partial_y \tau_{yz} = \gamma,\ \partial_x u + \partial_y v = 0,\\
(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2) = 6k_s^2,\\
\sigma_x - p = \lambda \partial_x u,\ \sigma_y - p = \lambda \partial_y v,\ \sigma_z - p = 0,\ 2\tau_{xy} = \lambda(\partial_x v + \partial_y u),\\
2\tau_{xz} = \lambda \partial_x w,\ 2\tau_{yz} = \lambda \partial_y w.
\end{gathered} \tag{6}$$

We make the change of variables in (6) by the following formulas

$$\sigma_x = k(\sqrt{3}\cos\omega + \sin\omega\cos 2\varphi),\ \sigma_y = k(\sqrt{3}\cos\omega - \sin\omega\cos 2\varphi),\ \tau_{xy} = k\sin\omega\sin 2\varphi. \tag{7}$$

Here $k = \delta k_s$, $0 < \delta < 1$ is some constant.

Substituting these relations into (6), we obtain

$$\begin{gathered}
(-\cos\omega + \sqrt{3}\sin\omega\cos 2\varphi)\partial_x \omega + \sqrt{3}\sin\omega\sin 2\varphi \partial_y \omega - 2\sin\omega \partial_y \varphi = 0,\\
(\cos\omega + \sqrt{3}\sin\omega\cos 2\varphi)\partial_y \omega + \sqrt{3}\sin\omega\sin 2\varphi \partial_x \omega + 2\sin\omega \partial_x \varphi = 0,\\
\partial_x \tau_{xz} + \partial_y \tau_{yz} = \gamma,\ \partial_x u + \partial_y v = 0,\ \tau_{xz}^2 + \tau_{yz}^2 = k_s^2 - k^2 = K^2,\\
\sigma_x - p = \lambda \partial_x u,\ \sigma_y - p = \lambda \partial_y v,\ \sigma_z - p = 0,\ 2\tau_{xy} = \lambda(\partial_x v + \partial_y u),\\
2\tau_{xz} = \lambda \partial_x w,\ 2\tau_{yz} = \lambda \partial_y w.
\end{gathered} \tag{8}$$

From (8) we see that the original system split into two subsystems: the first two equations essentially coincide with the equations describing the plane stress state

$$\begin{gathered}
(-\cos\omega + \sqrt{3}\sin\omega\cos 2\varphi)\partial_x \omega + \sqrt{3}\sin\omega\sin 2\varphi \partial_y \omega - 2\sin\omega \partial_y \varphi = 0,\\
(\cos\omega + \sqrt{3}\sin\omega\cos 2\varphi)\partial_y \omega + \sqrt{3}\sin\omega\sin 2\varphi \partial_x \omega + 2\sin\omega \partial_x \varphi = 0,
\end{gathered} \tag{9}$$

and equations reminiscent of the equations describing the plastic torsion of a rod, at $\gamma = 0$ and different yield point

$$\partial_x \tau_{xz} + \partial_y \tau_{yz} = \gamma,\ \tau_{xz}^2 + \tau_{yz}^2 = k_s^2 - k^2 = K^2. \tag{10}$$

Solving equations (9) and (10), it is possible to find the components of the stress tensor, while $\sigma_z = p = 1/2(\sigma_x + \sigma_y)$. The only problem is to determine the constant k.

To determine the components of the velocity vector, we obtain the following equations

$$\frac{\partial_x u}{2\sigma_x - \sigma_y} = \frac{\partial_y v}{2\sigma_y - \sigma_x} = \frac{\partial_x v + \partial_y v}{6\tau_{xy}},\quad \partial_x\Big(\frac{\partial_x w}{\sqrt{w_x^2 + w_y^2}}\Big) + \partial_y\Big(\frac{\partial_y w}{\sqrt{w_x^2 + w_y^2}}\Big) = \gamma.$$

Note that the last equation for $\gamma \neq 0$ has not yet been sufficiently studied, it has not even been included in the handbook [15].

As a result, it turned out that the solution of the system of equations (6) is actually a superposition of the plane stress state and plastic torsion around the *z* axis.

Let us give some other solutions of equations (6). For this, we find the group of point symmetries of the system of equations (6).

This group is generated by the operators

$$X_1 = \partial_x,\ \ X_2 = \partial_y,\ \ Y_i = \partial_{u_i},\ \ N = x\partial_x + y\partial_y,\ \ M = u_i\partial_{u_i},\ \ i = 1,2,3.$$
$$T = x_1\partial_{u_2} - x_2\partial_{u_1},\ \ Z_3 = x_1\partial_{x_2} - x_2\partial_{x_1} + u_1\partial_{u_2} - u_2\partial_{u_1},\quad S = \partial_p. \tag{11}$$

We are looking for a solution that is invariant under the subalgebra generated by the operators

$$X_1 + \alpha T + Y_1 + \beta Y_3 + \gamma S.$$

The solution should be sought in the form

$$u = x + \alpha xy + U(y), v = -\alpha/2x^2 + V(y),\ w = \beta x + W(y),\ p = \gamma x + P(y). \tag{12}$$

From the incompressibility equation we obtain

$$v = -\alpha/2(x^2 + y^2) - y + C_1.$$

Substituting (12) into (6) and we obtain the system of ordinary differential equations

$$\frac{d}{dy}(\lambda U') = \gamma,\ \ \frac{d}{dy}(\lambda W') = 0,\quad 6k_s^2\lambda^{-2} = 2(1+\alpha y)^2 + 6((U')^2 + \beta^2 + (W')^2). \tag{13}$$

From (13) we have

$$\lambda U' = \gamma y + C_2,\ \lambda W' = C_3.$$

Here $C_1,\ C_2,\ C_3$ – arbitrary constants.

We obtain

$$U'/W' = (\gamma y + C_2)/C_3,$$
$$6k_s^2\lambda^{-2} = 2(1+\alpha y)^2 + 6(\beta^2 + (1 + (C_4 + \nu y)^2)(W')^2),\ \ \nu = \gamma/C_3,\ \ C_4 = C_2/C_3.$$

The system of equations (13) can be reduced to quadratures, which are expressed through elliptic integrals.

The constructed solution can be used to describe the plastic flow of a layer compressed by rigid plates orthogonal to the oz axis.

Let us write system (6) in a cylindrical coordinate system $r, \theta, z.$

$$\partial_r\sigma_r + r^{-1}\partial_\theta\sigma_{r\theta} + (\sigma_r - \sigma_\theta)/r = 0,\ \partial_r\sigma_{r\theta} + r^{-1}\partial_\theta\sigma_\theta + 2\sigma_{r\theta}/r = 0,$$
$$\partial_r\sigma_{rz} + r^{-1}\partial_\theta\sigma_{\theta z} + \sigma_{rz}/r = 0,\ \sigma_r - p = \lambda\partial_r u,\ \ \sigma_\theta - p = \lambda(u/r + \partial_\theta v/r),\ \ \sigma_z - p = 0,$$
$$2\sigma_{r\theta} = \lambda(\partial_\theta u/r + r\partial_r(v/r)),\ \ 2\sigma_{z\theta} = \lambda r^{-1}\partial_\theta w,\ \ 2\sigma_{zr} = \lambda\partial_r w, \tag{14}$$
$$(\sigma_r - \sigma_\theta)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_\theta - \sigma_z)^2 + 6(\sigma_{r\theta}^2 + \sigma_{\theta z}^2 + \sigma_{rz}^2) = 6k_s^2,\ \partial_r u + u/r + \partial_\theta v/r = 0.$$

We are looking for an invariant solution on a subalgebra $Z_3 + M$, it has the form

$$u = U(\theta),\ \ v = V(\theta),\ \ w = W(\theta),\ p = P(\theta) + \alpha\ln r. \tag{15}$$

Substituting (15) into (14), we obtain the system of ordinary differential equations

$$d_\theta\sigma_r + (\sigma_r - \sigma_\theta) = \alpha,\ \ d_\theta\sigma_\theta + 2\sigma_{r\theta} = 0,\ \ d_\theta\sigma_{r\theta} + \sigma_{rz} = 0,\ \ \sigma_r - p = 0,$$
$$\sigma_\theta - p = \lambda(U/r + V'/r),\ \ \sigma_z - p = 0,\ \ 2\sigma_{r\theta} = \lambda(U'/r + r\partial_r(V/r)),$$
$$2\sigma_{z\theta} = \lambda r^{-1}W',\ \ 2\sigma_{zr} = \lambda/r, \tag{16}$$
$$(\sigma_r - \sigma_\theta)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_\theta - \sigma_z)^2 + 6(\sigma_{r\theta}^2 + \sigma_{\theta z}^2 + \sigma_{rz}^2) = 6k_s^2,$$
$$U + V' = 0.$$

Here the prime means the derivative with respect to θ.

From this we obtain the system of ordinary differential equations

$$(\lambda(U'-V))'=\alpha, \lambda(U'-V)=P, \quad (\lambda W')'=0, \quad 6k_s^2\lambda^{-2}=(U'-V)^2+(W')^2+1. \tag{17}$$

System (17) is solved completely similarly to system (13).

The found solution can be used to describe plastic flow in a converging flat channel with rigid and rough walls.

Other solutions of system of equations (1) can be found in [11].

Conclusion

In this work, a class of equations has been studied, which is called the equations describing the 3-dimensional deformed state. For these equations, a group of point symmetries is found, admitted by them in the sense of Lie. It is shown that a two-dimensional stress state is a superposition of a plane stress state and torsion around the *z* axis. Several invariant solutions of these equations are constructed.

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